Conformal quantum mechanics and Fick-Jacobs equation

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December 17, 2012

Abstract

It is found a relation between conformal quantum mechanics and Fick-Jacobs equation, which describes diffusion in channels. This relation is given between a family of channels and a family of conformal Hamiltonians. In addition, it is shown that a conformal Hamiltonian is associated with two channels with different geometry. Furthermore exact solutions for Fick-Jacobs equation are given for this family of channels.

1 Introduction

Recently, mathematical techniques developed in an area has been employed to study systems from other different areas. In this subject, an amazing result is given by AdS_{d+1}/CFT_d duality, which allows a relation between $(d+1)$-dimensional gravitational theory and certain classes of d-dimensional Yang-Mills theories [1]. The conformal group is very important in this duality, in fact this group is the the largest symmetry group of special relativity

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[2]. Now, the Schrödinger group is a non-relativistic conformal group [3, 4]. This last group is the symmetry group for the free Schrödinger equation and has been important to study non-relativistic AdS_{d+1}/CFT_d duality [5, 6]. To study AdS_2/CFT_1 correspondence, the so call conformal quantum mechanics has been proposed as CFT_1 dual to AdS_2, see [7, 8]. The conformal quantum mechanics is invariant under Schrödinger group and has been employed to study problems from black-holes to atomic physics [9, 10, 11]. Furthermore, the simplest model of diffusion is described by the Fick equation and Sophus Lie showed that this equation is invariant under Schrödinger group [14]. Other studies about diffusion phenomena and Schrödinger group can be seen in [15]. Then, the conformal symmetry, relativistic or non-relativistic, is very important to understand diverse aspects of different systems.

Now, when the diffusion is in a channel, which has the shape of surface of revolution with cross sectional area $A(x)$, the Fick equation has to be changed to Fick-Jacobs equation [17]

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[D_0 A(x)\frac{\partial}{\partial x}\left(\frac{C(x,t)}{A(x)}\right)\right], \tag{1}$$

where $C(x,t)$ is the particle concentration and D_0 is the diffusion coefficient. This last equation is important to study diffusion in biological channels or zeolites [18, 19, 20, 21, 22, 23, 24]. The Fick-Jacobs equation does not look like the free Schrödinger equation, but it can be mapped to Schrödinger equation with an effective potential [25].

In this paper we will show that the Fick-Jacobs equation is equivalent to conformal quantum mechanics for a family of channels. Then for this family of channels the Fick-Jacobs equation is invariant under Schrödinger group. Also, it is found that the equivalence is given between a family of channels and a family of conformal Hamiltonians. In addition, it is shown that a conformal Hamiltonian is associated with two channels with different geometry. For these channels an exact solution for the Fick-Jacobs equation is given.

This paper is organized in the following way: in section 2 a brief review about Schrödinger group and conformal quantum mechanics is given; in section 3 it is shown that the Fick-Jacobs equation is equivalent to conformal quantum mechanics for a set of particular channels and an exact solution for this equation is given. Finally, in section 4 a summary is given.

2 Schrödinger group

The free Schrödinger equation

$$i\hbar\frac{\partial\psi\left(\vec{x},t\right)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi\left(\vec{x},t\right), \tag{2}$$

is invariant under the following transformation: Galileo transformation $x_i'=x_i+v_it$, rotations $x_i'=R_{ij}x_j$, space-time translation $t'=t+a, x_i'=x_i+x_{0i}$, anisotropic scaling $t'=b^2t, \quad x_i'=bx_i$ and special conformal transformation [3, 4]

$$t'=\frac{t}{1+at}, \qquad x_i'=\frac{x_i}{1+at}. \tag{3}$$

Some work about Schrödinger group and conformal symmetry can be seen in [26, 27, 28, 29, 30, 31, 32, 33].

Now, the Schrödinger equation for the 1-dimensional conformal quantum mechanics is given by

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=H\psi(x,t), \qquad H=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{g}{x^2}, \tag{4}$$

which is invariant under Schrödinger transformation. The classical system with the potential $V(r)=gr^{-2}$ was first studied by Jacobi [34] and the quantum system was proposed by Jackiw [12]. Using the Schrödinger group generators, the spectrum of Hamiltonian (4) was found by de Alfaro, Fubini and Furlan [16]. The Hamiltonian (4) appears in different contexts, from black-holes to atomic physics [9, 10, 11]. In the next section we will show that this systems also appears in diffusion phenomena.

3 Conformal quantum mechanics and Fick-Jacobs equation

Using $C(x,t)=\sqrt{A(x)}\psi(x,t)$, the Fick-Jacobs equation becomes

$$\frac{\partial\psi(x,t)}{\partial t}=\left[D_0\frac{\partial^2}{\partial x^2}-\frac{D_0}{2\sqrt{A(x)}}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{A(x)}}\frac{\partial A(x)}{\partial x}\right)\right]\psi(x,t). \tag{5}$$

Then, if we propose $\psi(x,t) = e^{-Et}\phi(x)$, we get the following Schrödinger equation

$$E\phi(x) = H\phi(x), \tag{6}$$

where

$$H = -D_0\frac{\partial^2}{\partial x^2} + \frac{D_0}{2\sqrt{A(x)}}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{A(x)}}\frac{\partial A(x)}{\partial x}\right). \tag{7}$$

Now, the family of channels with cross sectional area $A(x) = ax^{2\nu}$ is associated with the following family of Hamiltonians

$$H = -D_0\frac{\partial^2}{\partial x^2} + \frac{g}{x^2}, \qquad g = D_0\nu\left(\nu - 1\right). \tag{8}$$

For each ν we have a conformal quantum mechanics Hamiltonian (4). However, for each Hamiltonian (8) we have two channels, namely each Hamiltonian is associated with two ν values. For example, $\nu = 0$ and $\nu = 1$, represent different sectional areas, but both cases give the same Hamiltonian

$$H = -D_0\frac{\partial^2}{\partial x^2}. \tag{9}$$

The solution for the Schrödinger equation (6) with the Hamiltonian (8) is given by

$$\phi_\nu(x) = |x|^{\frac{1}{2}}J_{\pm\left(\frac{2\nu-1}{2}\right)}\left(\pm\sqrt{\frac{E}{D_0}}x\right), \tag{10}$$

where $J_p\left(w\right)$ is the Bessel function of order p. Then, if the channel has cross sectional area $A(x) = ax^{2\nu}$, the solution for the Fick-Jacobs is given by

$$C_\nu(x,t) = Be^{-Et}|x|^{\frac{2\nu+1}{2}}J_{\pm\left(\frac{2\nu-1}{2}\right)}\left(\pm\sqrt{\frac{E}{D_0}}x\right), \tag{11}$$

here B is a constant.

Notice that whether $\nu = 0$ the solution

$$C_{\nu=0}(x,t) = e^{-Et}\left(B_1\sin\left(\sqrt{\frac{E}{D_0}}x\right) + B_2\cos\left(\sqrt{\frac{E}{D_0}}x\right)\right), \tag{12}$$

is obtained. While if $\nu = 1$, the solution

$$C_{\nu=1}(x,t) = e^{-Et}|x| \left(B_1 \sin\left(\sqrt{\frac{E}{D_0}}x\right) + B_2 \cos\left(\sqrt{\frac{E}{D_0}}x\right)\right) \tag{13}$$

is gotten. We can see that $\nu = 0$ and $\nu = 1$ are associated with the same Hamiltonian, but the particle concentration is not the same.

4 Summary

In this paper we shown a relation between conformal quantum mechanics and Fick-Jacobs equation. This relation is given between a family of channels and a family of conformal Hamiltonians. It was found that a conformal Hamiltonian is associated with two channels with different geometry. In addition, exact solutions for Fick-Jacobs equation are given for this family of channels. This result is interesting, because the conformal quantum mechanics has been proposed as a realization of AdS_2/CFT_1 duality and Fick-Jacobs equation is employed to describe diffusion in biological channel. Then, it is possible that mathematical techniques from string theory can be employed to study some biological problems.

Acknowledgments

We would like to thank P.A Horvathy for his comments about the history of Schrödinger symmetry.

References

[1] J. Maldacena, *The Large N limit of Superconformal Field Theories and Supergravity,* Adv Theor. Math. Phys. 2, 231 (1998), [hep-th/9711200].

[2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, *Large N field theories, string theory and gravity,* Phys.Rept. 323 (2000) 183-386, [hep-th/9905111].

[3] C. R. Hagen, *Scale and conformal transformations in galilean-covariant field theory,* Phys. Rev. D **5** 377 (1972).

[4] U. Niederer, *The maximal kinematical invari- ance group of the free Schrodinger equation,* Helv. Phys. Acta **45** (1972) 802.

[5] D. T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry,* Phys. Rev. D **78**, 046003 (2008), [arXiv:0804.3972 [hep- th]].

[6] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs,* Phys. Rev. Lett. 101, 061601 (2008), [arXiv:0804.4053 [hep-th]].

[7] C. Chamon, R. Jackiw, S. -Y. Pi and L. Santos, *Conformal quantum mechanics as the CFT1 dual to AdS2,* Phys. Lett. B **701**, 503 (2011), [arXiv:1106.0726 [hep-th]].

[8] R. Jackiw, S.-Y. Pi, *Conformal Blocks for the 4-Point Function in Conformal Quantum Mechanics,* Phys.Rev. D **86**, (2012) 045017, Erratum-ibid. D86 (2012) 089905, [arXiv:1205.0443 [hep-th]]

[9] R. Britto-Pacumio, J. Michelson, A. Strominger, A. Volovich, *Lectures on superconformal quantum mechanics and multiblack hole moduli spaces,* Contributed to NATO Advanced Study Institute on Quantum Geometry. Published in Progress in string theory and M-theory, 235-264 (1999), [hep-th/9911066].

[10] H. E. Camblong, C. R. Ordonez, *Anomaly in conformal quantum mechanics: From molecular physics to black holes,* Phys. Rev. D **68**, 125013 (2003), [hep-th/0303166].

[11] K. Sakamoto, K. Shiraishi, *Conformal Quantum Mechanics in Two Black Hole Moduli Space,* Phys.Rev. D **66** (2002) 024004, [arXiv:hep-th/0203152].

[12] R. Jackiw, *Introducing Scale Symmetry,* Phys. Today **25** (1), 23 (1972).

[13] V. de Alfaro, S.Fubini and G. Furlan, *Conformal Invariance in Quantum Mechanics,* Nuovo Cim. **34** A, 569 (1976).

[14] S. Lie, Arch. Math. Nat. vid. (Kristiania) 6, 328 (1882).

[15] S. Stoimenov, M. Henkel, *Dynamical symmetries of semi-linear Schrdinger and diffusion equations,* Nucl. Phys. B723, 205-233 (2005), [arXiv:math-ph/0504028].

[16] V. de Alfaro, S.Fubini and G. Furlan, *Conformal Invariance in Quantum Mechanics,* Nuovo Cim. **34** A, 569 (1976).

[17] M. Jacobs, *Diffusion Processes,* Springer, New York, (1967).

[18] D. Reguera, J. M. Rubí, *Kinetic equations for diffusion in the presence of entropic barriers,* Phys. Rev. E **64**, 061106 (2001).

[19] D. Reguera, G. Schmid, P. S. Burada, J. M. Rubí, P. Reimann, P. Hänggi *Entropic Transport: Kinetics, Scaling, and Control Mechanisms,* Phys. Rev. Lett. **96**, 130603 (2006).

[20] P. S. Burada, P. Hänggi, F. Marchesoni, G. Schmid, P. Talkner, *Diffusion in Confined Geometries,* Chem. Phys. Chem. **10**, 45 (2009).

[21] A. M. Berezhkovskii, M. A. Pustovoit, S. M. Bezrukov, *Diffusion in a tube of varying cross section: Numerical study of reduction to effective one-dimensional description,* J. Chem. Phys. **126**, 134706 (2007).

[22] I. Pineda, M. V. Vázquez, L. Dagdug, *Diffusion between Two Chambers Connected by a Conical Capillary,* in *New trends in statistical physics* Edited by A. Macias, L. Dagdug, (World Scientific, Singapore 2010) pp. 147-162.

[23] R. Zwanzig, *Diffusion past an entropy barrier,* **96**, J. Chem. Phys.3926 (1992).

[24] P. Kalinay, J. K. Percus, *Corrections to the Fick-Jacobs equation,* Phys. Rev. E **74**, 041203 (2006).

[25] J. M. Romero, O. González-Gaxiola, G. Chacón-Acosta, *Exact solution to Fick-Jacobs equatio,* to appears in International Journal of Pure and Applied mathematics (IJPAM), Volume 82, No. 1 (2013), [arXiv:1108.1164 [math-ph]] .

[26] C. Duval, P. A. Horvathy, *Non-relativistic conformal symmetries and Newton-Cartan structures,* J. Phys. A **42**, 465206 (2009), [arXiv:0904.0531 [math-ph]].

[27] C. Duval, G. W. Gibbons, P. Horvathy, *Celestial mechanics, conformal structures and gravitational waves,* Phys. Rev. D **43** 3907-3922 (1991), [arXiv:hep-th/0512188].

[28] C. Duval, M. Hassaine, P. A. Horvathy, *The geometry of Schrödinger symmetry in non-relativistic CFT,* Ann. Phys. **324** 1158 (2009), [arXiv:0809.3128 [hep-th]].

[29] C. Leiva, M. S. Plyushchay, *Conformal symmetry of relativistic and non-relativistic systems and Ads/CFT correspondence,* Annals Phys. **307** (2003) 372-391, [arXiv:hep-th/0301244].

[30] C. Leiva, M. S. Plyushchay, *Superconformal mechanics and nonlinear supersymmetry,* JHEP **0310** 069 (2003), [arXiv:hep-th/0304257].

[31] A. Anabalon, M. S. Plyushchay, *Interaction via reduction and non-linear superconformal symmetry,* Phys.Lett. B **572** 202-209 (2003), [arXiv:hep-th/0306210].

[32] A. Galajinsky, *Remark on quantum mechanics with conformal Galilean symmetry,* Phys. Rev. D **78** 087701 (2008), [arXiv:0808.1553 [hep-th]].

[33] A. Galajinsky, I.V. Masterov, *Remark on quantum mechanics with N=2 Schrödinger supersymmetry,* Phys. Lett. B **675** 116 (2009), [arXiv:0902.2910 [hep-th]].

[34] C. G. J. Jacobi, *Vorlesungen uber Dynamik,* Univ. Königsberg 1842-43.